Quantum Computation by Decoherence Free Operator Algebras

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The states of the physical algebra, namely the algebra generated by the measurements performed in encoding and processing qbits, are considered instead of those of the whole system-algebra. If the physical algebra is DF – that is it commutes with the interaction Hamiltonian, and the system Hamiltonian is the sum of arbitrary terms either commuting with or belonging to the physical algebra – then its states are DF. One of the considered examples shows that the smallest number of physical qbits encoding a DF logical qbit is reduced from four to three.

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Environment induced decoherence [1,2] is the main obstruction to the physical viability of quantum computing [3]. To overcome this obstacle, quantum error correcting codes have been devised [4,3]. Besides these *active* methods, where decoherence is controlled by repeated application of error correction procedures, a more recent *passive* approach has emerged, where logical qbits are encoded in decoherence free (DF) subspaces [5]- [10]. In them coherence is protected by the peculiar structure of the coupling Hamiltonian.

So far the notion of a DF state has been considered within the total Hilbert space of the considered system, namely with reference to the whole operator algebra of the system, whereas a more physical approach consists in confining the consideration to the space of the states on the physical algebra, that is the operator algebra involved in encoding and manipulating qbits. The characterization of such state spaces corresponds to the construction of the irreducible representations of the aforementioned C^* algebra [12]. Quantum computing without active error correcting codes requires the use of physical algebras admitting DF irreducible representations, which therefore will be called DF algebras. The construction of such representations is performed here by showing that suitable factorizations of the total Hilbert space exist, where entanglement with the environment (or equivalently decoherence, once this is traced out) is confined to only one factor, the other factor carrying an irreducible representation of the DF algebra.

This more physical approach leads to a fruitful generalization of the notion of a DF state. It is shown for instance that, for a generic uniform coupling of an array of physical qbits to an arbitrary environment, while the conventional notion of DF space requires at least four physical qbits to encode a logical one [11], three are enough in this new setting.

Consider the dynamics of a system S coupled to a bath B, the *universe* evolving unitarily under the Hamiltonian $H = H_S \otimes \mathbf{1}_B + \mathbf{1}_S \otimes H_B + H_I$, where H_S and H_B denote respectively the system and the bath Hamiltonian, H_I the interaction Hamiltonian, $\mathbf{1}_S$ and $\mathbf{1}_B$ the identity operators on the Hilbert space $\mathcal{H}_S$ of the system and $\mathcal{H}_B$ of the bath respectively. Let $\mathcal{A}_S \equiv gl(\mathcal{H}_S)$ denote the operator algebra of $\mathcal{H}_S$, (which for simplicity is assumed to be finite dimensional) and $\mathcal{A}_{DF}$ the invariant subalgebra of $\mathcal{A}_S$ consisting of operators commuting with H_I:

$$[\mathcal{A}_{DF}, H_I] = 0. \tag{1}$$

As a subalgebra of $\mathcal{A}_S$, $\mathcal{A}_{DF}$ has a natural C^* algebra structure, by which, if measurements on the system are confined to those represented by operators in $\mathcal{A}_{DF}$, state spaces can be identified with its irreducible representations. (Given a state of $\mathcal{A}_{DF}$ the corresponding representation can be built in general by the GNS construction [12].) A pure state of $\mathcal{A}_{DF}$, namely a state prepared by a complete set of measurements of $\mathcal{A}_{DF}$, such remains under time evolution, if the system Hamiltonian is the sum of an operator belonging to $\mathcal{A}_{DF}$, giving rise to unitary evolution, and an operator that commutes with $\mathcal{A}_{DF}$, which for such a state gives rise to no evolution at all.

To be specific consider an array of N qbits. Let σ_1, σ_2, σ_3 be the usual Pauli matrices and σ_0 denote the 2×2 identity matrix. If these matrices are intended to be, as usual, representations of pseudospin Hermitian operators in the single qbit state space, the operator algebra for the whole array is generated by

$$M(i_1, i_2, ..., i_N) \doteq \bigotimes_{j=1}^{N} \sigma_{i_j}; \quad i_j = 0, 1, 2, 3. \tag{2}$$

Let

$$S_i = \frac{1}{2} \sum_{j=1}^{N} M(i\delta_{1j}, i\delta_{2j}, ..., i\delta_{Nj}) \tag{3}$$

denote the total pseudospin and assume, as frequently done in the literature [7], a uniform collective coupling to the environment

$$H_I = \sum_{i=1}^{3} S_i B_i, \tag{4}$$

where the bath operators B_i commute with $\mathcal{A}_S$ and then with $\mathcal{A}_{DF}$. As to the system Hamiltonian, under the usual hypothesis of equivalent uncoupled qbits [7]

$$H_s = \varepsilon S_3, \tag{5}$$

it commutes with $\mathcal{A}_{DF}$, which, as said above, avoids decoherence of states of $\mathcal{A}_{DF}$, even with the possible addition of terms belonging to $\mathcal{A}_{DF}$, like scalar couplings

$$\sum_{i=1}^{3} M(i_1 = 0, i_2 = 0, ..., i_j = i, ..., i_k = i, ...i_N = 0) \tag{6}$$

due to the exchange interaction present in NMR computing [13]. Let $\mathcal{A}_E$ denote the algebra generated by the *errors* S_i. Of course $\mathcal{A}_{DF} \cap \mathcal{A}_E$ is generated by (the identity and by) the Casimir operator

$$S^2 = \sum_{i=1}^{3} S_i^2, \tag{7}$$

by which, in order to factor the operator algebra as a product of such subalgebras, the state space must be reduced to an S^2 eigenspace. To this end the system Hilbert space $\mathcal{H}_S$, as the tensor product of N fundamental representations of $sl(2, C)$, can be decomposed as the sum of irreducible representations of the algebra $sl(2, C)$ generated by the operators S_i:

$$\mathcal{H}_S = \bigoplus_j \bigoplus_{k=1}^{n_j} \mathcal{D}_j, \tag{8}$$

where the indices j fix the values of the Casimir operator: $S^2\mathcal{D}_j = j(j+1)\mathcal{D}_j$.

The operator algebra of the generic eigenspace of S^2 can be identified with the product of the representations of the DF and the error algebras on $\bigoplus_{k=1}^{n_j} \mathcal{D}_j$:

$${}_j\mathcal{A}_S \equiv gl\left(\bigoplus_{k=1}^{n_j} \mathcal{D}_j\right) \sim \ {}_j\mathcal{A}_{DF} \otimes \ {}_j\mathcal{A}_E. \tag{9}$$

In fact the S^2 eigenspace in its turn can be identified with the direct product of an n_j dimensional complex space and just one copy of the irreducible representation

$$\bigoplus_{k=1}^{n_j} \mathcal{D}_j \sim C^{n_j} \otimes \mathcal{D}_j \tag{10}$$

through the one to one correspondence $|k,m\rangle \leftrightarrow |k\rangle \otimes |m\rangle$, where $|k,m\rangle$ denotes the eigenvector of S_3 with eigenvalue m in the kth copy of $\mathcal{D}_j$, while $|m\rangle$ denotes the only such eigenvector in $\mathcal{D}_j$ and $|k\rangle$ is the kth element of a basis of C^{n_j}. To be more precise, once the mutually orthogonal vectors $|k,j\rangle$ are fixed, one defines $|k,m\rangle \equiv (S_-)^{j-m}\ |k,j\rangle$ by means of the lowering operator $S_- = S_1 - iS_2$.

Since the generic operator O on C^{n_j} gives through this identification an operator $O \otimes \mathbf{1}_{\mathcal{D}_j}$ on $\bigoplus_{k=1}^{n_j} \mathcal{D}_j$ commuting with $\mathcal{A}_E$, which is generated by operators of the form $\mathbf{1}_{C^{n_j}} \otimes Q$, and since all operators can be realized in terms of the operators $M(i_1, i_2, ..., i_N)$, it follows that operators on C^{n_j} can be identified with (equivalence classes of) elements of $\mathcal{A}_{DF}$. This proves that the generic S^2 eigenspace can be identified with the product of two spaces, carrying irreducible representations of $\mathcal{A}_{DF}$ and $\mathcal{A}_E$ respectively. It should be stressed that coherent superpositions of S^2 eigenstates with different eigenvalues do not exist as states of $\mathcal{A}_{DF}$, as they live in different representations.

As a first example of a qbit array, collectively and uniformly coupled to the environment, consider a system of three physical qbits. The corresponding DF algebra is generated by

$$b_{23} \doteq \sum_{j=1}^{3} 1 \otimes \sigma_j \otimes \sigma_j, \quad b_{31} \doteq \sum_{j=1}^{3} \sigma_j \otimes 1 \otimes \sigma_j, \quad b_{12} \doteq \sum_{j=1}^{3} \sigma_j \otimes \sigma_j \otimes 1, \tag{11}$$

Since the factorization of ${}_j\mathcal{A}_S$ in Eq. (9) is trivial for $S^2 = 15/4$, as the error algebra generates the whole operator algebra, the analysis is confined to the eigenspace $\mathcal{H}_{1/2}$ with $S^2 = 3/4$. Using the symbol ${}_{1/2}O$ for the representation of the generic operator O in $\mathcal{H}_{1/2}$, for instance it can be checked that, if one defines the invariant operator

$$E_{123} \doteq \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k, \tag{12}$$

with ε_{ijk} denoting the usual completely antisymmetric symbol, the $\mathcal{H}_{1/2}$ representations of invariant operators

$${}_{1/2}\tau_1 = (\ {}_{1/2}b_{12} - \ {}_{1/2}b_{23})/\sqrt{12}$$

$${}_{1/2}\tau_2 = \ {}_{1/2}E_{123}\sqrt{12}$$

$${}_{1/2}\tau_3 = (\ {}_{1/2}b_1 - 2\ {}_{1/2}b_2 + \ {}_{1/2}b_3)/6 \tag{13}$$

are the generators of an $su(2)$ algebra,

$$\left[\ {}_{1/2}\tau_i,\ {}_{1/2}\tau_j\right] = 2i\sum_{k=1}^{3}\varepsilon_{ijk}\ {}_{1/2}\tau_j, \tag{14}$$

with the Casimir given by

$${}_{1/2}\tau^2 \equiv \sum_{j=11/2}^{3} \tau_j^2 = 3\hat{\mathbf{1}}. \tag{15}$$

The corresponding universal enveloping algebra $\mathcal{A}\left({}_{1/2}\tau\right)$, which coincides with ${}_{1/2}\mathcal{A}_{DF}$, is then the operator algebra of a two state system and the total operator algebra ${}_{1/2}\mathcal{A}$ is given by the product of this algebra and the universal enveloping algebra $\mathcal{A}\left({}_{1/2}\mathbf{S}\right)$ of the total pseudospin algebra:

$${}_{1/2}\mathcal{A} = \mathcal{A}\left({}_{1/2}\tau\right) \otimes \mathcal{A}\left({}_{1/2}\mathbf{S}\right) = \ {}_{1/2}\mathcal{A}_{DF} \otimes \mathcal{A}\left({}_{1/2}\mathbf{S}\right), \tag{16}$$

as a particular instance of Eq. (9). As a consequence the state space $\mathcal{H}_{1/2}$ can be identified with the tensor product of two two-dimensional representation spaces $\mathcal{H}_{1/2}(\tau)$ and $\mathcal{H}_{1/2}(\mathbf{S})$ respectively of $\mathcal{A}\left({}_{1/2}\tau\right)$ and $\mathcal{A}\left({}_{1/2}\mathbf{S}\right)$:

$$\mathcal{H}_{1/2} = \mathcal{H}_{1/2}(\tau) \otimes \mathcal{H}_{1/2}\left(\mathbf{S}\right), \tag{17}$$

which coincides with Eq. (10) for $j = 1/2$ and $n_j = 2$. According to what has been illustrated above, this factorization has far reaching physical consequences: if all measurement processes are limited to (Hermitian) elements of $\mathcal{A}_{DF}$, then a state $\rho = |\psi\rangle\langle\psi| \otimes \rho_{\mathbf{S}}$, which is the product of a pure state in $\mathcal{H}_{1/2}(\tau)$ and an arbitrary density matrix in $\mathcal{H}_{1/2}\left(\mathbf{S}\right)$, is a pure state of the physical algebra $\mathcal{A}_{DF}$. If in particular the initial state has this structure (possibly with $\rho_{\mathbf{S}}$ being itself a pure state of $\mathcal{A}\left({}_{1/2}\mathbf{S}\right)$, this corresponding to an arbitrary pure state in $\mathcal{H}_{1/2}$), then, in spite of the decoherence of $\rho_{\mathbf{S}}$ (or equivalently the entanglement with the environment if this is not traced out) produced by the coupling of the environment to the pseudospin operators, the state maintains phase coherence as to the physical algebra, which is then DF. This means that the considered three qbit array encodes a DF logical qbit, compared to the four qbits needed within the conventional approach [11].

As a further example, consider now a four qbit array. In this case, while the factorization is trivial and useless for the $S^2 = 6$ $(j = 2)$ representation, it is still trivial but fruitful for the carrier space $\mathcal{H}_0$ of the two degenerate $S^2 = 0$ representations, where it gives rise to the DF states already considered in the literature. To be specific it can be checked that the $\mathcal{H}_0$ representations of invariant operators

$$ {}_0\tau_1 \doteq (\ {}_0b_{14} + \ {}_0b_{23} - \ {}_0b_{12} - \ {}_0b_{34})/(4\sqrt{3}),$$

$$ {}_0\tau_2 \doteq (\ {}_0E_{234} + \ {}_0E_{124} - \ {}_0E_{134} - \ {}_0E_{123})/(8\sqrt{3}),$$

$$ {}_0\tau_3 \doteq -(\ {}_0b_{14} + \ {}_0b_{12} + \ {}_0b_{13}2)/3 \tag{18}$$

obey the same relations as their analogues in Eq.s (14,15), whose enveloping algebra once again is the operator algebra of a DF logical qbit. As represented in $\mathcal{H}_0$ the DF subalgebra coincides with the total operator algebra, the representation of the total pseudospin algebra being the trivial (scalar) one.

For the four qbit array, apart from the reproduction of a DF qbit of vanishing pseudospin, the present approach gives also rise to a DF qtrit. Consider in fact the 9-dimensional $S^2 = 2$ ($j = 1$) eigenspace $\mathcal{H}_1$ containing three degenerate 3-dimensional representations. It can be checked, for instance, that the $\mathcal{H}_1$ representation of invariant operators

$$
{}_1\tau_1 \doteq \ {}_1E_{134}/(-2\sqrt{3}),
$$

$$
{}_1\tau_2 \doteq (\ {}_1E_{134} - 3\ {}_1E_{124})/(4\sqrt{6}),
$$

$$
{}_1\tau_3 \doteq (\ {}_1E_{234} + \ {}_1E_{123})/(4\sqrt{2}) \tag{19}
$$

obey the usual commutation rules of $su(2)$ generators as in Eq. (14), while ${}_1\tau^2 \equiv \sum_{j=1}^{3} \tau_j^2 = 8\hat{\mathbf{1}}$. In this case the 9-dimensional state space ${}_1\mathcal{H}$ can be identified with the product of the 3-dimensional irreducible representations of the DF algebra and the total pseudospin algebra. In perfect analogy to what said for the three qbit array one can arrange in the considered $S^2 = 2$ eigenspace a DF qtrit, namely a tridimensional state space of the DF algebra. Of course in this case the whole representation algebra ${}_1\mathcal{A}_{DF}$ cannot be produced by linear combination of the $sl(2, C)$ generators (and the identity) only, but products of two of them must be included too.

In conclusion what has been shown can be of use both with reference to the considered examples and more generally as a method to identify for given systems several alternative DF spaces, which can give rise to more chances for finding physically viable realizations of quantum computing. In particular the possibility to test DF qbit encoding in arrays of just three physical qbits may represent a substantial bonus in the near future.

More generally the approach in terms of representations of DF algebras may shed some light on the physical relevance of quantum coherence, which in principle, due to the structure of the Hamiltonian, could be present in unexpected situations if system algebras can be factored as the product of uncoupled collective algebras, one of them decoupled from the environment too.

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